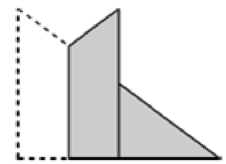
3rd Annual Math Match – February 13, 2012

1. Prove that for every $n \in \mathbb{N}$, $n! \leq \left(\frac{n+1}{2}\right)^n$.

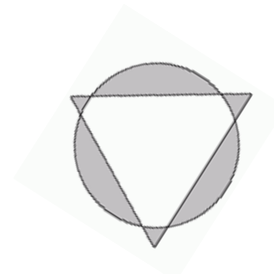
2. Consider a triangular sheet of paper with vertices at the points (0,0), (4.0), and (0,3). By making a single vertical crease in the paper and then folding the left portion of the triangle over the crease so that it overlaps the right portion we obtain a polygon with five sides, coloured grey. Find the smallest possible area of this resulting polygon.



3. Let f be a continuous function on $[0,1]$, differentiable on $(0,1)$, and such that $f(1) = 0$. Show that for some $c \in (0,1)$,

$$\frac{f(c)}{c} = -f'(c).$$

4. You are given an equilateral triangle whose sides have length 1 and a circle of radius r centered at the centroid of the triangle. What value of r will minimize the total area of the region consisting of those points which either lie inside the circle but outside the triangle or inside the triangle but outside the circle?



5. Is the number $\frac{2^{58}+1}{5}$ prime or composite?

6. A car travels downhill at 72 mph, on the level at 63 mph, and uphill at only 56 mph. The car takes 4 hours to travel from town A to B. The return trip takes 4 hours and 40 minutes. Find the distance between the two towns.

7. Show that, for each $n \geq 3$, $n!$ can be represented as the sum of n distinct divisors of itself. For example, $3! = 1 + 2 + 3$.

8. You find a treasure map. It says on it:

> *The treasure of Captain Bird is buried on the Island of the Parrot. Near the centre of this island three great trees form a triangle. The mightiest of the three is a great oak older than the treasure itself. Towards the west of the oak, some distance away there stands an elm tree, and towards the east of the oak there stands an ash. To find the treasure of Captain Bird count out the paces from the oak to the elm. When you get to the elm make a precise left turn and count out the same number of paces. Mark this spot with a flag. Return to the oak and count out the paces from the oak to the ash. When you get to the ash make a precise right turn and count out the same number of paces. Mark this spot with a flag. The treasure lays buried midway between the two flags.*

So, you rent a boat a head off to the Island of the Parrot only to discover that there is no oak tree. You can locate the elm and ash, but the oak tree is gone. Where is the treasure? *(Notice: "precise turn" means 90° turn)*

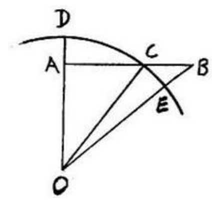
Solutions or hints:

1. $\sqrt[n]{n!} \le \frac{1+2+\cdots+n}{n} = \frac{\cancel{n}(n+1)}{2\cancel{n}}$ $(GM \le AM)$ so $n! \le \left(\frac{n+1}{2}\right)^n$

2. Maximize the "double-overlap" trapezoid. If the fold is at x, then this area is $\left(3-\frac{9}{8}x\right)x$, and it's maximum occurs at $x=\frac{4}{3}$ and equals 2. Hence the minimum area of the desired shape is $6-2=4$.

3. Consider the function $g(x)=xf(x)$. Since $f(1)=0$ then $g(0)=g(1)=0$ and from Rolle's Theorem there is a $c \in (0,1)$ s.t. $g'(c)=0$. So $0=g'(c)=f(c)+cf'(c)$ and finally $\frac{f(c)}{c}=-f'(c)$.

4. Consider the upper right part of the figure. On the diagram, $AB=.5$ and $OA \perp AB$. We have $OA=\frac{1}{2}\cot\frac{\pi}{3}$. Let $OA=h$ and $AC=a$. Then $r^2=a^2+h^2$, where $h=\frac{\sqrt{3}}{6}$ from properties of equilateral triangle.



Let $\angle AOC=\alpha$ and $\angle COB=\beta$. Then $\alpha=\tan^{-1}\left(\frac{a}{h}\right)$ and $\beta=\frac{\pi}{3}-\alpha$.
Area of $DAC=\frac{1}{2}\alpha r^2-\frac{ah}{2}$ and area of $CBE=\left(\frac{1}{2}-a\right)\frac{h}{2}-\frac{1}{2}\left(\frac{\pi}{3}-\alpha\right)r^2$.
$\frac{d(total\ area)}{da}=2a\tan^{-1}\left(\frac{a}{h}\right)-\frac{\pi a}{3}=0$ for $=\frac{h}{\sqrt{3}}$, so $r=\left(\frac{h}{\sqrt{3}}\right)^2+h^2$, but $h=\frac{\sqrt{3}}{6}$, so $r=\frac{1}{3}$.

5. $\frac{2^{58}+1}{5}$ is an integer as $2^{58}=2^2(2^4)^{14}\equiv 4\cdot 1^{14}\equiv 4(mod\ 5)$ hence $2^{58}+1\equiv 0(mod\ 5)$.
Now $2^{58}+1=(2^{29}+1)^2-(2^{15})^2=(2^{29}+2^{15}+1)(2^{29}-2^{15}+1)$. Both factors are greater than 5, so $2^{58}+1=5ab$, where a and b are integers greater than 1. Therefore $\frac{2^{58}+1}{5}$ is composite.

6. Let the total distance travelled from *A* to *B* downhill, on the level, and uphill be *x*, *y*, and *z* respectively. Then comparing the time, we have $\begin{cases}\frac{x}{72}+\frac{y}{63}+\frac{z}{56}=4\\ \frac{x}{56}+\frac{y}{63}+\frac{z}{72}=\frac{14}{3}\end{cases}$ which after multiplying by $LCD=7\cdot 8\cdot 9$ becomes $\begin{cases}7x+8y+9z=4\cdot 7\cdot 8\cdot 9\\ 9x+8y+7z=\frac{14}{3}\cdot 7\cdot 8\cdot 9\end{cases}$. After adding, we have $16(x+y+z)=\left(\frac{26}{3}\right)\cdot 7\cdot 8\cdot 9$, therefore the distance between *A* and *B* = $x+y+z=273$ miles.

7. We will use induction. The base step is shown for $n=3$: $3!=1+2+3$.
Assume inductive hypothesis, that we have a representation for $n=k$, with 1 as one of the divisors.
That means, we have $k!=1+d_2+\cdots+d_k$, where $d_2<\cdots<d_k$ are the distinct divisors of *k*.
Then $(k+1)!=(k+1)+(k+1)d_2+\cdots+(k+1)d_k=1+k+(k+1)d_2+\cdots+(k+1)d_k$, which is the sum of $k+1$ distinct divisors of $(k+1)!$, with 1 as one of the divisors.
Therefore, by induction, for each $n\ge 3$, $n!$ can be represented as the sum of *n* distinct divisors of itself.

8. Consider system of coordinates such that $\overline{EA}$ is horizontal, and assume that *O* is North of $\overline{EA}$. Let $EA=a$; the horizontal distance OE_H from *O* to *E* be *x*; the vertical distance OE_V be *y*; the flags be at *F* and *G*; and the treasure be at *T*. Then $OA_H=a-x=AG_V$, $EF_H=y=AG_H$, and $EF_V=x$, as on diagram. Now we can see that the midpoint *T* of *FG* is horizontally $y+\frac{a-2y}{2}=\frac{a}{2}$ units from *E*, and $x+\frac{a-x-x}{2}=\frac{a}{2}$ units from *A*. Therefore *T* is on the perpendicular bisector of $\overline{EA}$, half the distance EA from the midpoint of $\overline{EA}$, in either direction (up or down).