Math 312 - AB1 Assignment # 1 Solutions

1. SOLUTION: Let's say that $\vec{v} = 2\hat{\imath} + 3\hat{\jmath}$ goes from point A (tip of vector $\vec{a}$) to point B (tip of vector $\vec{b}$). Since the midpoint of AB is the tip of $\vec{m} = 2\hat{\imath} + \hat{\jmath}$,

$$\begin{aligned} \vec{a} &= \vec{m} - 0.5\vec{v} = \hat{\imath} - 0.5\hat{\jmath} \\ \vec{b} &= \vec{m} - 0.5\vec{v} = 3\hat{\imath} + 2.5\hat{\jmath} \end{aligned}$$

So $A = (1, -\frac{1}{2})$ and $B = (3, \frac{5}{2})$

2. SOLUTION: Since the ring isn't accelerating, the forces must satisfy

$$\vec{F_1} + \vec{F_2} + \vec{F_3} + \vec{F_4} = \vec{0} .$$

Since $\vec{F_1} = 4\hat{\imath}$, $\vec{F_2} = 5\hat{\jmath}$, and $\vec{F_3} = -8\hat{k}$,

$$\vec{F_4} = -4\hat{\imath} - 5\hat{\jmath} + 8\hat{k}$$

Thus the magnitude of $\vec{F_4}$ is $\sqrt{(-4)^2 + (-5)^2 + 8^2} = \sqrt{105} \approx 10.25$ (Newtons).
The direction angles can then be computed as follows:

$$\begin{aligned} \alpha &= \cos^{-1}\left(\frac{-4}{\sqrt{105}}\right) \approx 112.98^\circ \\ \beta &= \cos^{-1}\left(\frac{-5}{\sqrt{105}}\right) \approx 119.21^\circ \\ \gamma &= \cos^{-1}\left(\frac{8}{\sqrt{105}}\right) \approx 38.67^\circ \end{aligned}$$

3. SOLUTION:

   (a) L_1 has "on-ramp" $1\hat{\imath} + 1\hat{k}$ and direction vector $2\hat{\imath} - 3\hat{\jmath}$, so its equations are:

$$\frac{x-1}{2} = \frac{y-0}{-3} \ , \ z = 1$$

   L_2 has "on-ramp" $4\hat{\jmath} - 2\hat{k}$ and direction vector $6\hat{\imath} + 3\hat{\jmath} + \hat{k}$, so its vector form is:

$$\vec{R} = (6\hat{\imath} + 3\hat{\jmath} + \hat{k})t + (4\hat{\jmath} - 2\hat{k})$$

   (b) It's easiest to use their parametric forms, and equate the x, y, and z positions. But we need to use a different parameter for each line, say s and t.

$$\begin{aligned} 2t + 1 &= x = 6s &(1) \\ 3t &= y = 3s + 4 &(2) \\ 1 &= z = s - 2 &(3) \end{aligned}$$

   From equation (3), we must have $s = 3$.
   Use this in equation (2) to get $t = 13/3$.
   Then the left side of equation (1) says $x = 26/3 + 1 = 29/3$.
   But the right side of equation (1) says $x = 18$.
   Thus no s and t satisfy the three equations at once, so there is no intersection.

(c) One normal to the plane is $\vec{n} = 3\hat{\imath} + 2\hat{\jmath} - 6\hat{k}$. This vector has length $|\vec{n}| = 7$, so a unit vector normal to the plane is $\hat{n} = \frac{3}{7}\hat{\imath} + \frac{2}{7}\hat{\jmath} - \frac{6}{7}\hat{k}$.

(d) To solve for the intersection point of P and L_2, we can substitute the parametric equations for L_2 into the equation for P:

$$\begin{aligned} 3(6s) + 2(3s+4) - 6(s-2) &= 11 \\ 18s + 6s + 8 - 6s + 12 &= 11 \\ 18s &= -9 \quad \text{, so } s = -\tfrac{1}{2} \end{aligned}$$

The corresponding point is then $(-3, 2.5, -2.5)$, by substituting the s-value back into the parametric form of L_2.

The angle at which they cross is the complement of the acute angle θ that the direction vector of L_2 makes with the normal of P. (These were found in parts (a) and (c).)

$$\theta = \arccos\left(\frac{(6\hat{\imath} + 3\hat{\jmath} + \hat{k}) \cdot (3\hat{\imath} + 2\hat{\jmath} - 6\hat{k})}{|6\hat{\imath} + 3\hat{\jmath} + \hat{k}||3\hat{\imath} + 2\hat{\jmath} - 6\hat{k}|}\right) = \arccos\left(\frac{18}{(\sqrt{46})(7)}\right) \approx 67.72^\circ$$

(Notice that θ is acute, so we did use the correct direction of the normal.) So the plane and line cross at an angle of

$$90^\circ - \arccos\left(\frac{18}{(\sqrt{46})(7)}\right) \approx 22.28^\circ$$

(e) L_1 is parallel with P because its direction vector is perpendicular to the normal of P:

$$(2\hat{\imath} - 3\hat{\jmath}) \cdot (3\hat{\imath} + 2\hat{\jmath} - 6\hat{k}) = 0$$

The distance between them is the length of the projection of any bridge $\vec{b}$ (from L_1 to P) onto any vector normal to P. Our bridge will be the vector from the point $(1, 0, 1)$ on L_1 to the point $(3, 1, 0)$ on P:

$$\vec{b} = 2\hat{\imath} + 1\hat{\jmath} - 1\hat{k}$$

The projection onto $\vec{n}$ (from part (c)) is then

$$proj_{\vec{n}}(\vec{b}) = \left(\frac{\vec{b} \cdot \vec{n}}{\vec{n} \cdot \vec{n}}\right)\vec{n} = \left(\frac{14}{(7)(7)}\right)\vec{n} = \left(\frac{2}{7}\right)\vec{n}$$

The length of the the projection is then $\left(\frac{2}{7}\right)|\vec{n}| = 2$.

4. SOLUTION:

$$\begin{aligned} \vec{v}_{||} &= proj_{\vec{q}}(\vec{v}) = \left(\frac{\vec{v} \cdot \vec{q}}{\vec{q} \cdot \vec{q}}\right)\vec{q} = \left(\frac{-20}{2}\right)(\hat{\jmath} + \hat{k}) = -10\hat{\jmath} - 10\hat{k} \\ \vec{v}_{\perp} &= \vec{v} - \vec{v}_{||} = 5\hat{\jmath} - 10\hat{\jmath} + 10\hat{k} \end{aligned}$$

Quick check: $\vec{v}_{||}$ is a multiple of $\vec{q}$, and $\vec{v}_{\perp} \cdot \vec{q} = 0$.

5. SOLUTION: We need $\vec{A}\cdot(\vec{C}-s\vec{A}-t\vec{B})=0$ and $\vec{B}\cdot(\vec{C}-s\vec{A}-t\vec{B})$. Expanding each equation using the given vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, we get

$$\begin{aligned} 1(2-s-2t)+1(-1-s+t)+2(4-2s-t) &= 0 \text{ and} \quad (4)\\ 2(2-s-2t)-1(-1-s+t)+1(4-2s-t) &= 0 \quad (5)\end{aligned}$$

These simplify to $9-6s-3t=0$ and $9-3s-6t=0$, from which the solution $s=t=1$ is easily seen.

6. SOLUTION:

$$(\hat{i}+2\hat{j}-\hat{k})\times(r\hat{i}+s\hat{j}+4\hat{k})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & -1\\ r & s & 4\end{vmatrix}=(8+s)\hat{i}-(4+r)\hat{j}+(s-2r)\hat{k}$$

For this to be the zero vector, we need $s=-8$ and $r=-4$.

7. SOLUTION:

(a) $|\vec{AD}|=\sqrt{(1-3)^2+(1-1)^2+(1-0)^2}=\sqrt{5}$. Using "on-ramp" $\vec{A}$ and direction vector $\vec{AD}$, we get

- vector parametric equation: $\vec{r}=(3\hat{i}+\hat{j})+(-2\hat{i}+1\hat{k})t\ ,\ t\in R$
- equations form: $\dfrac{x-3}{-2}=z\ ,\ y=1$

(b) Let $\vec{u}$ be the vector from A to B and let $\vec{v}$ be the vector from A to C. The area of the parallelogram with adjacent sides $\vec{u}$ and $\vec{v}$ is

$$|\vec{u}\times\vec{v}|=\left\|\begin{matrix}\hat{i} & \hat{j} & \hat{k}\\ -3 & -1 & 6\\ -2 & -1 & 2\end{matrix}\right\|=|4\hat{i}-6\hat{j}+1\hat{k}|=\sqrt{16+36+1}=\sqrt{53}$$

so the area of the triangle with vertices A, B, C is $\frac{1}{2}\sqrt{53}$.

We have just found a normal vector to the plane (the cross product). Using the point $A(3,1,0)$ in the plane (or use B or C), we can build an equation for the plane:

$$4(x-3)-6(y-1)+1(z-0)=0$$

(c) Let $\vec{w}$ be the vector from A to D. The tetrahedron volume is one-sixth the volume of the parallelepiped with vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ as adjacent edges, so

$$Volume=\left(\frac{1}{6}\right)|\vec{w}\cdot(\vec{u}\times\vec{v})|=\left(\frac{1}{6}\right)|(-2\hat{i}+1\hat{k})\cdot(4\hat{i}-6\hat{j}+1\hat{k})|=\left(\frac{1}{6}\right)|-7|=\frac{7}{6}$$

8. SOLUTION: Use the first identity given in 1.14 for each term:

$$\begin{aligned} LHS &= \hat{i}\times(\hat{i}\times\vec{A})+\hat{j}\times(\hat{j}\times\vec{A})+\hat{k}\times(\hat{k}\times\vec{A})\\ &= [(\hat{i}\cdot\vec{A})\hat{i}-(\hat{i}\cdot\hat{i})\vec{A}]+[(\hat{j}\cdot\vec{A})\hat{j}-(\hat{j}\cdot\hat{j})\vec{A}]+[(\hat{k}\cdot\vec{A})\hat{k}-(\hat{k}\cdot\hat{k})\vec{A}]\\ &= [(\hat{i}\cdot\vec{A})\hat{i}-\vec{A}]+[(\hat{j}\cdot\vec{A})\hat{j}-\vec{A}]+[(\hat{k}\cdot\vec{A})\hat{k}-\vec{A}]\\ &= [(\hat{i}\cdot\vec{A})\hat{i}+(\hat{j}\cdot\vec{A})\hat{j}+(\hat{k}\cdot\vec{A})\hat{k}]-3\vec{A}\\ &= \vec{A}-3\vec{A}\\ &= -2\vec{A}\ =\ RHS\end{aligned}$$