Lemma 2. For non-negative constants $c \leq g \leq h$, we have

$$\int_g^h z^2 \cos^{-1} \sqrt{\frac{z^2- c^2}{z^2}}\ dz$$

$$= \begin{cases} [(h^3 \sin^{-1} \frac{c}{h} - g^3 \sin^{-1} \frac{c}{g}) - c^3(\frac{g\sqrt{g^2- c^2}- h\sqrt{h^2- c^2}}{2\ c^2} + \frac{1}{2} \ln \frac{h-\sqrt{h^2- c^2}}{g-\sqrt{g^2- c^2}})]/3 \text{ if } c > 0, \\ 0 \text{ if } c = 0. \end{cases}$$

Proof: For $c > 0$, we let $\sin\theta = c/z$ and use integration by parts to get

$$\int z^2 \cos^{-1} \sqrt{1 - (\frac{c}{z})^2}\ dz = \frac{z^3}{3} \sin^{-1} \frac{c}{z} - \frac{c^3}{3} \int \frac{1}{\sin^3\theta}\ d\theta.$$

Using the fact that

$$\int \frac{1}{\sin^3\theta}\ d\theta = -\frac{\cos\theta}{2\sin^2\theta} + \frac{1}{2} \ln\tan\frac{\theta}{2}$$

and the half-angle formulae for sine and cosine, we obtain the result after the indefinite integral is evaluated at g and h. For $c = 0$, the result is trivial.

Lemma 3. For non-negative constants $d \leq g \leq h$, we have

$$\int_g^h z^2 \cos^{-1} \frac{d}{z}\ dz$$

$$= \begin{cases} [(h^3 \cos^{-1} \frac{d}{h} - g^3 \cos^{-1} \frac{d}{g}) - d^3(\frac{h\sqrt{h^2- d^2}- g\sqrt{g^2- d^2}}{2\ d^2} + \frac{1}{2} \ln \frac{h+\sqrt{h^2- d^2}}{g+\sqrt{g^2- d^2}})]/3 \text{ if } d > 0, \\ \pi(h^3 - g^3)/6 \text{ if } d = 0. \end{cases}$$

Proof: For $d > 0$, we let $\cos\theta = d/z$ and follow the same procedure as shown in the proof of Lemma 2. However, we now need to deal with the integral $\int 1/\cos^3\theta\ d\theta$ instead of $\int 1/\sin^3\theta\ d\theta$. This can be done by using the fact that

$$\int \frac{1}{\cos^3\theta}\ d\theta = \frac{\sin\theta}{2\cos^2\theta} + \frac{1}{2} \ln(\sec\theta + \tan\theta).$$