UCFV Math Club
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Optimal Broadcasting Strategies

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A Brief Tour of LaTeX

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Part I
Optimal Broadcast Strategies

- Introduction to the Model
- Broadcast Strategies
- An Optimization Problem
- A Miniature Example
- Deterministic Strategy Solution
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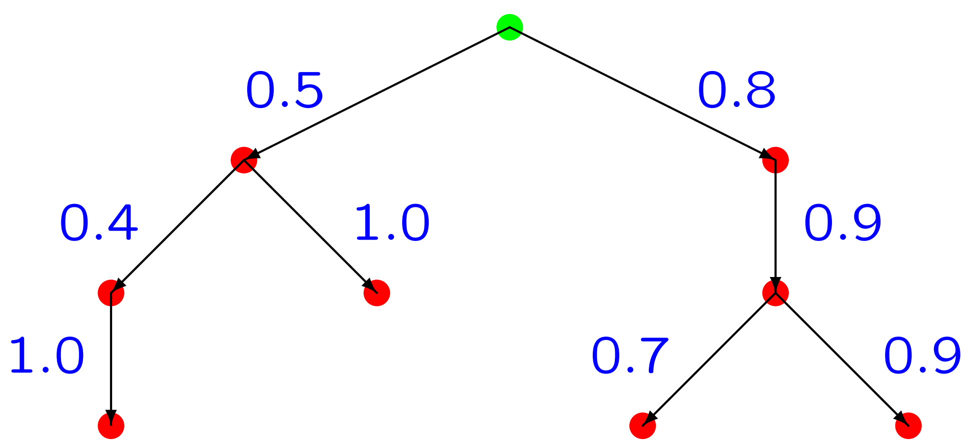
Introduction to the Model

Imagine a tree-shaped network structure where only the root node is aware of a message which must be sent to all other nodes.



Suppose further that each communication line succeeds on each transmission attempt with some known probability , and that any sending node is not aware of success or failure of its attempts.

Broadcast Strategies

There are two ways in which one might instruct the nodes to perform the broadcast.

- A **Deterministic strategy** assigns to each parent node a **sequence** in which it should make calls to its children.

  For example, the sequence provided to the root node might be

  $$L, R, L, L, R, L, L, R, L, \ \ldots$$

- A **Randomized strategy** assigns to each parent node a **probability distribution** by which it should choose which child to call at each opportunity.

  For example, the probability distribution provided to the root node might be

  $$\text{Prob}(L) = 0.6 \quad , \quad \text{Prob}(R) = 0.4$$

Rules of the Game

- At time zero, the root node begins its attempts to inform its children.

- Each call attempt takes one round (say, one second).

- When a successful transmission is made to any node, it begins calls to its own children (if it has any) at the start of the next round.

- Nodes are never aware as to which (if any) of their call attempts have succeeded.

- The broadcast is complete when every node knows the message.

An Optimization Problem

One natural way of measuring the performance of a strategy is the expected time required for the broadcast to finish.

Some questions...

- (Statistics) Given a broadcast strategy, how can we determine its expected completion time?

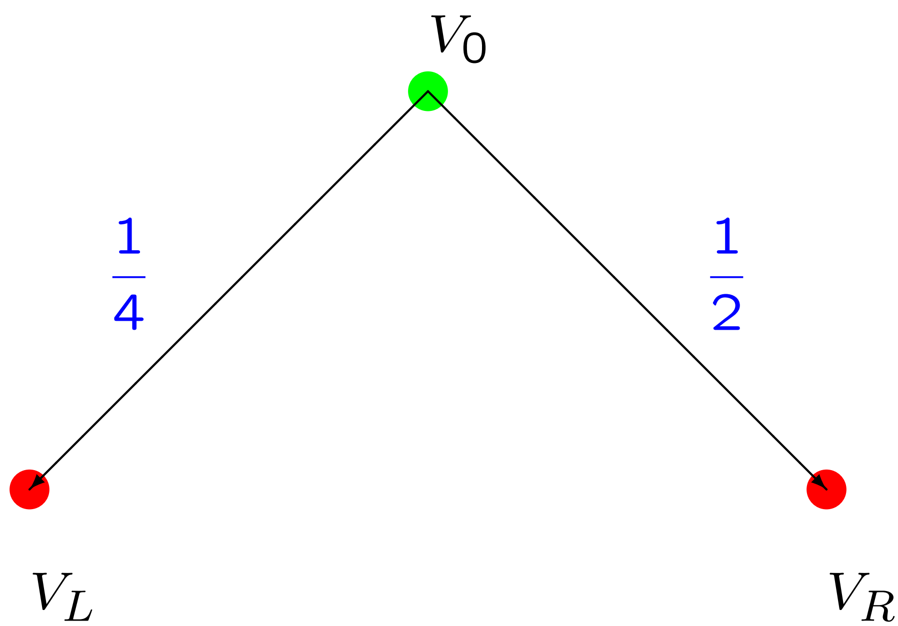
- (Optimization) How can we find an optimal broadcast strategy for a given network?

- (Stats and Opt) How can we find the optimal broadcast time for a given network?

- (Graph Theory) Which tree graph(s) with n nodes have minimum broadcast time?

A Miniature Example

Consider the following example - the smallest interesting case.



Remember - the blue numbers represent probability of successful completion of each call made.

We should expect optimal strategies attempt calls to the left more often.

Optimal Deterministic Strategy

To find an optimal deterministic strategy, build a sequence which maximizes the probability of completion following each round.

The probability P_n of completion following the n^{th} call is a function of the number $k \leq n$ of calls made to the left.

$$\begin{aligned} P_n &= \text{Pr}(V_L \text{ hears within } k \text{ calls}) \\ &\qquad * \text{Pr}(V_R \text{ hears within } n-k \text{ calls}) \\ &= \left(1 - \left(1 - \frac{1}{4}\right)^k\right) * \left(1 - \left(1 - \frac{1}{2}\right)^{n-k}\right) \end{aligned}$$

One can easily program Maple to determine the particular k that maximizes P_n for each n.

However, the corresponding expected time to broadcast is not easy to find exactly.

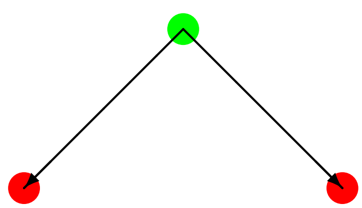
Randomized Strategies

Randomized broadcast strategies are much more fun to measure and optimize.

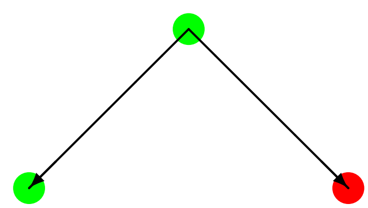
Under any fixed randomized strategy, the broadcast process is a Markov Chain, whose states are the possible subsets of informed nodes.

In our miniature example, there are 4 states:

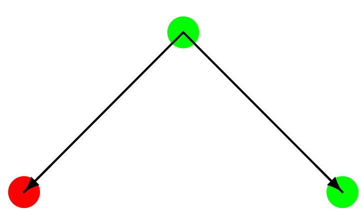
Initial State S_1



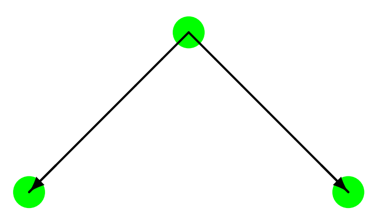
Left-awake State S_2



Right-awake State S_3



Final State S_4



Transition Probabilities

Informally, a process which is in exactly one state at each time step is called a Markov Chain if the probability distribution of which state is next depends only on the current state.

In our example, if calls are made according to

$$\Pr(\text{Left}) = p \quad , \quad \Pr(\text{Right}) = 1 - p,$$

then the transition probabilities are:

Potential Next State

Current State	S_1	S_2	S_3	S_4
S_1	$1 - \frac{p}{4} - \frac{1-p}{2}$	$\frac{p}{4}$	$\frac{1-p}{2}$	0
S_2		$1 - \frac{1-p}{2}$		$\frac{1-p}{2}$
S_3			$1 - \frac{p}{4}$	$\frac{p}{4}$
S_4				1

The Transient and Fundamental Matrices

Delete the last row and column in the table above to form the matrix T of transient (non-final) states:

$$T = \begin{bmatrix} 1-\frac{p}{4}-\frac{1-p}{2} & \frac{p}{4} & \frac{1-p}{2} \\ 0 & 1-\frac{1-p}{2} & 0 \\ 0 & 0 & 1-\frac{p}{4} \end{bmatrix}$$

The matrix $N = (I-T)^{-1}$ is called the fundamental matrix of the Markov Chain:

$$N = \begin{bmatrix} \frac{4}{2-p} & \frac{2p}{(1-p)(2-p)} & \frac{8(1-p)}{p(2-p)} \\ 0 & \frac{2}{1-p} & 0 \\ 0 & 0 & \frac{4}{p} \end{bmatrix}$$

Completion Time Moments

Let the random variable t_i represent the time to reach the final state S_4, beginning in state S_i, $1 \leq i \leq 3$.

For $k \geq 1$, let $\boldsymbol{m}^{(\boldsymbol{k})}$ be the vector of moments

$$\boldsymbol{m}^{(\boldsymbol{k})} = \begin{bmatrix} E[t_1^k] \\ \\ E[t_2^k] \\ \\ E[t_3^k] \end{bmatrix}$$

Theorem 1 *(adapted from Theorem 3.1 of [3]) All moments of each random variable t_i are finite, and are related by the recursion*

$$\boldsymbol{m}^{(\boldsymbol{k})} = N \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + T \sum_{j=1}^{k-1} \binom{k}{j} \boldsymbol{m}^{(\boldsymbol{k}-\boldsymbol{j})} \right)$$

Optimal Expected Broadcast Time

Theorem 1 is a much more powerful tool than we need right now!

It does tell us that the expected time to reach state S_4 from S_1 is

$$E[t_1] = \frac{1}{2-p}\left(4 + \frac{2p}{1-p} + \frac{8(1-p)}{p}\right)$$

Minimizing this expression with $p \in (0,1)$ yields

$$p^* = 1 - \sqrt{\frac{2}{3}\sqrt{3} - 1} \approx .60668$$

The optimal expected broadcast time is therefore

$$\frac{4\sqrt{3} - 4)}{p^*(1-p^*)(2-p^*)} \approx 8.8073$$

Further Work

1. Find easily computable bounds on optimal broadcast time (deterministic and randomized) based on network structure.

2. Find easily computable heuristic strategies which broadcast in time provably close to optimal.

3. Find the tree design on n nodes which admits the fastest broadcast.

Bibliography

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