Problems for Math Match 2019 with solutions

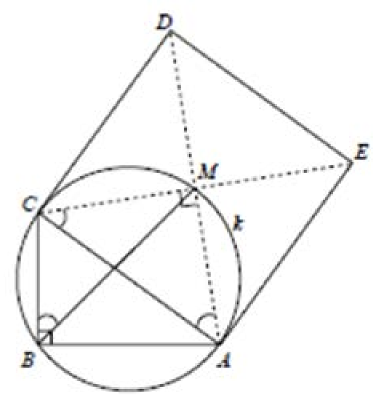
1. Let $n > 3$ be a positive integer. Equilateral triangle ABC is divided into n^2 smaller congruent equilateral triangles (with sides parallel to its sides). Let m be the number of rhombuses that contain two small equilateral triangles and d the number of rhombuses that contain eight small equilateral triangles. Find the difference $m - d$ in terms of n.

Solution:

m must be $\frac{3n(n-1)}{2}$ and d must be $\frac{3(n-2)(n-3)}{2}$, so their difference is $6n - 9$.

2. Let ABC be a right triangle with right angle at B. Let $ACDE$ be a square drawn exterior to triangle ABC. If M is the center of this square, find the measure of $\angle MBC$.

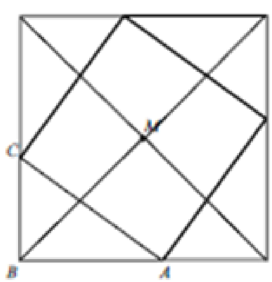
Solution I:



$\triangle MCA$ is a right isosceles triangle with $\angle ACM = 90°$ and $\angle MAC = 45°$. Since $\angle ABC = 90°$, there is a circle k with diameter AC which also passes through B and M. As inscribed angles, $\angle MAC = \angle ABC$. Thus, the measure of $\angle MBC = 45°$.

Solution II:

Place 3 copies of triangle ABC on the square, as shown below. The diagonals of the large square meet at M, so $\angle MBC = 45°$.



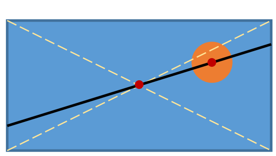
3. Two runners start running laps at the same time, from the same starting position. Sue runs a lap in 50 seconds while George runs a lap in 30 seconds. When will the runners next be side by side? Generalize your answer for the problem when Sue runs a lap in m seconds while George runs a lap in n seconds.

Answer:

75 seconds; $\frac{mn}{m-n}$ (*Hint:* $\left(\frac{1}{30} - \frac{1}{50}\right) t = 1$; generally $\left(\frac{1}{n} - \frac{1}{m}\right) t = 1$)

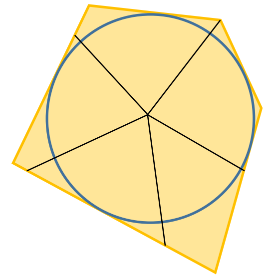
4. Two friends go to a pizza place regularly where a rectangular pizza is served. This pizza is called the hole in one pizza because the chef always cuts out a circular piece of any radius randomly from the rectangular pizza. How can the two friends equally divide the pizza with one cut?

Solution:



5. You have a cake in a shape of a circumscribed n-polygon (a convex polygon that contains an inscribed circle). How would you divide the cake equally among five people so that each person receives the same amount of icing?

Solution:



Mark 5 points on the edges of the cake so that the distance between the consecutive points measured along the perimeter is $\frac{1}{5}$ of the perimeter. Make cuts from the centre of the circle to the points marked on the edges.

6. 101 wise men stand in a circle. Each of them either thinks that the Earth orbits Jupiter or that Jupiter orbits the Earth. Once a minute, all the wise men express their opinion at the same time. Right after that, every wise man who stands between two people with a different opinion from him changes his opinion himself. The rest do not change. Prove that at one point they will all stop changing opinions. After what time we can guarantee that the wise man will no longer change their opinion.

Solution:

If the opinions are a and b, the circle consists of a concatenation of strings of the form aaa ..., bbb ..., and ... $abab$ ... (both starting and ending with either a or b). Any person that is part of one of the first two types of string will never change their opinion. Any person on the end of a string of the third type will also not change their opinion, and every person in the middle of the third string will change their opinion, thereby decreasing the length of the string of the third type (by exactly 2, unless the length of the string is 3, in which case it disappears). So, the whole process will end after at most 50 minutes.

7. Show that if 101 different integers are chosen from 1 to 200, inclusive, thare must be 2 with the property that one is divisible by the other.

Solution:

Represent each of the 101 integers x_i as $2^{k_i}a_i$, where a_i is odd. Now $1 \leq x_i \leq 200$, and $1 \leq a_i \leq 199$ for all i. There are only 100 odd integers from 1 to 199. So, $a_i = a_j$ for some $i \neq j$. Suppose $k_i < k_j$, then x_i divides x_j.

8. Show that, for any integer n, the number $n^3 - 9n + 27$ is not divisible by 81.

Solution:

Assume $81 \mid n^3 - 9n + 27$ for contr. Then $3 \mid n^3 - 9n + 27$, so $3 \mid n^3 \Rightarrow n = 3k$ which means $n^3 - 9n + 27 = 27(k^3 - k + 1)$. But $k^3 - k + 1 \equiv 1 \not\equiv 0 \ (mod\ 3)$ as $3 \mid k^3 - k = k(k-1)(k+1)$, so contradiction.