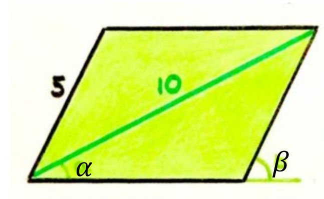
Problems for the 7th Annual Math Match 2020

1. Show that among any 37 integers that are not divisible by 7, there are 7 integers with the sum divisible by 7.

2. Show that 24 divides $p^2 - 1$ for any prime $p > 3$.

3. Refer to the diagram. Angles α and β are complementary. What is the area of the parallelogram?

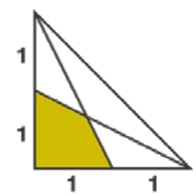


4. A circle is inscribed in an isosceles trapezoid with bases a and b. Find the area of this circle.

5. Show that $\sqrt{3 - 2\sqrt{2}} + \sqrt{6 - \sqrt{32}}$ is rational.

6. There are 9 checkers positioned in one of the 3x3 corners of an 8x8 chessboard. You can move a checker by applying central symmetry with respect to any other checker as long as the destination square is empty. Using a finite number of such moves, can you transfer all the checkers to a different 3x3 corner of the chessboard?

7. What is the area of the region of the right triangle, shaded in yellow in the diagram below?



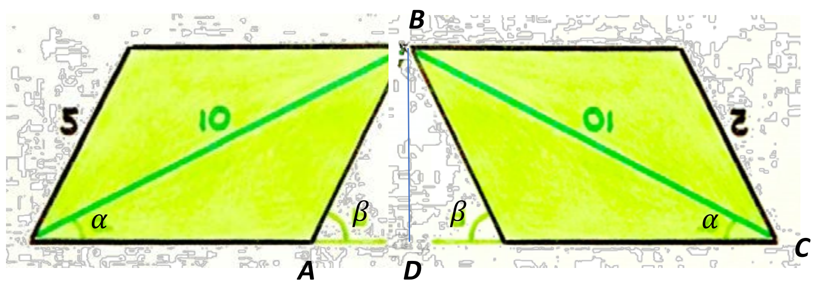
8. Is it possible to divide the lines in the plane into pairs of perpendicular lines so that every line belongs to exactly one pair?

Solutions:

1. Numbers that are not divisible by 7 may have up to 6 different remainders. By pigeonhole principle, there are 7 numbers with the same remainder from division by 7. So, the sum of these 7 numbers is divisible by 7.

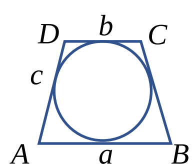
2. Since $p > 3$ is prime than $p - 1$ and $p + 1$ are two consecutive even numbers. So one of them divides by 4 and their product divides by 8. Among the three consecutive numbers $p - 1, p, p + 1$, one is divisible by 3, but p is not divisible by 3, so one of $p - 1$ or $p + 1$
Must be divisible by 3. Thus, $p^2 - 1 = (p - 1)(p + 1)$ is divisible by $8 \cdot 3 = 24$.

3.



Area of $\Delta ABC = 25$, as $AB \perp BC$.
$\Delta ABC \sim \Delta ADB$
If $AD = x$, then $BD = 2x$. So, $x^2 + (2x)^2 = 25$, which gives $x = \sqrt{5}$.
Thus, Area of $2 \cdot \Delta ADB = \sqrt{5} \cdot 2\sqrt{5} = 10$.
So area of the parallelogram is $2 \cdot (25 - 10) = 30$

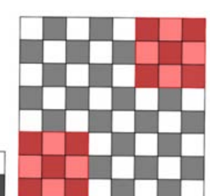
4. Since the trapezoid, call it $ABCD$, is cyclic, then $|AB| + |CD| = |BC| + |DA|$. So, if c is the length of the arm of the trapezoid, then $a + b = 2c$ which means that $= \frac{a+b}{2}$.

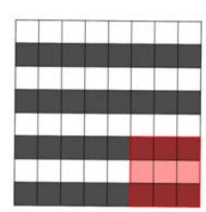


From Pythagorean equation, we have $h^2 = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 = ab$,
so $h = \sqrt{ab}$ and consequently $r = \frac{\sqrt{ab}}{2}$.
Thus, the area of the circle is $\frac{ab\pi}{4}$.

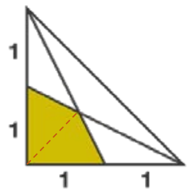
5. $\sqrt{3 - 2\sqrt{2}} + \sqrt{6 - \sqrt{32}} = \sqrt{2 - 2\sqrt{2} + 1} + \sqrt{4 - 4\sqrt{2} + 2} = \sqrt{\left(\sqrt{2} - 1\right)^2} + \sqrt{\left(2 - \sqrt{2}\right)^2} =$
$\left|\sqrt{2} - 1\right| + \left|2 - \sqrt{2}\right| = \sqrt{2} - 1 + 2 - \sqrt{2} = 1$

6. Observe that the move does not change the colour of the square. Assume we start from the left top corner. We cannot move them to the top right or bottom left corner as they have different number of squares of the same colour.
The same justification works for the right bottom square, if we colour the board as on the picture.





7. Notice that areas of all the small triangles with base 1 are the same as their height is the same (see the diagram). Denote this area by x. Then, the area of the long triangle by the diagonal is $2x$. Since the area of the original triangle is 2, then $6x = 2$ and $x = \frac{1}{3}$. Thus, the area of the shaded region is $\frac{2}{3}$.



8. Form families consisting of all mutually parallel lines. Put into a group two families whose lines are perpendicular. For each group, choose an arbitrary line l not parallel to either family. Each line in a family intersects exactly one point of l, and each point of l lies on exactly one line in the family. Thus each point of l defines one line from each family, and these two lines form a pair. This procedure may be applied to all groups, so that every line in the plane is in exactly one pair.