Problems for the 8th Annual Math Match 2023

1. Create a 4-digit number with different digits taken from the set $\{1, 2, \dots, 9\}$. Then, find the sum of all such 4-digit numbers.

2. Consider a set of n different lines in a plane such that each pair of lines intersects but no three lines intersect at the same point. Show that the lines cut the plane into $\frac{1}{2}(n^2 + n + 2)$ parts.

3. Show how to find the sum

$$2 + 22 + 222 + \cdots + \underbrace{222\dots2}_{2022\ times}$$

4. Show that any convex polygon can be enclosed by a rectangle with area not larger than twice the area of this polygon.

5. Find the area of a cyclic octagon with four consecutive sides of length 1 and the remaining four sides of length 2.

6. What factor of the form $i!$ should we remove from 1!2!3!...99!100! to make a perfect square number?

7. Find all integers n, $1 \leq n \leq 300$ for which n^n is a perfect cube number. How many of such numbers are there?

8. Find all polynomials $P(x)$ satisfying the condition $xP(x-1) = (x-2)P(x)$ for all real x.

Solutions:

1. Observe that the number of all such 4-digit numbers with different digits is $9 \cdot 8 \cdot 7 \cdot 6 = 3024$. For every number a, we can find a number b such that the sum of corresponding digits is equal to 10. For example, if $a = 1234$ than $b = 9876$. The number of all such pairs is ½ $\cdot 9 \cdot 8 \cdot 7 \cdot 6 = 1512$. The sum of numbers in each pair is 11110 (check the example). So the total sum of all 4-digit numbers with different digits taken from {1, 2, … , 9} is $1512 \cdot 11110 = \mathbf{16798320}$.

2. By induction, one line cuts the plane into 2 parts, which agrees with the formula, as $\frac{1}{2}(1^2 + 1 + 2) = 2$. Assume that k lines, $l_1, l_2, \ldots, l_k$, cut the plane into $\frac{1}{2}(k^2 + k + 2)$ parts. We need to show that $k + 1$ lines, $l_1, l_2, \ldots, l_k, l_{k+1}$ satisfying the conditions of the problem cut the plane into $\frac{1}{2}((k+1)^2 + k + 1 + 2)$ parts. Line l_{k+1} meets the lines $l_1, l_2, \ldots, l_k$ at the corresponding points $A_1, A_2, \ldots, A_k$, which are all different. These points cut the line l_{k+1} into $k + 1$ sections. Each of these sections lies in one of the parts of the plane created by the lines $l_1, l_2, \ldots, l_k$ and cuts this part into 2 parts. So, the number of parts of the plane created by $l_1, l_2, \ldots, l_k, l_{k+1}$ lines is the number of parts created by $l_1, l_2, \ldots, l_k$ lines, increased by $k + 1$. Since

$\frac{1}{2}(k^2 + k + 2) + k + 1 = \frac{1}{2}(k^2 + k + 2 + 2k + 2) = \frac{1}{2}((k+1)^2 + k + 1 + 2)$, the statement is proven.

3. Let

$$S = 2 + 22 + 222 + \cdots + \underbrace{222\ldots2}_{2022\ times}$$

Then

$$\frac{9}{2}S = 9 + 99 + 999 + \cdots + \underbrace{999\ldots9}_{2022\ times}$$

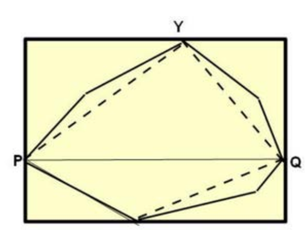
$$= 10 - 1 + 100 - 1 + 1000 - 1 + \cdots + \underbrace{100\ldots0}_{2022\ zeros} - 1$$

$$= 10 + 100 + 1000 + \cdots + \underbrace{100\ldots0}_{2022\ zeros} - 2022$$

$$= 10\frac{1 - 10^{2022}}{1 - 10} - 2022 = \frac{10^{2023} - 10 - 9 \cdot 2022}{9} = \frac{10^{2023} - 18208}{9}$$

Therefore, $S = \mathbf{\frac{2(10^{2023} - 18208)}{81}}$.

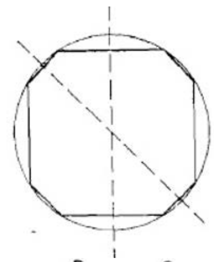
4. Consider the longest segment contained in the polygon (or one of the longest segments if there are more than one such segments). This will be one of the segments connecting two vertices, say P and Q. Let p and q be the two lines perpendicular to $\overline{PQ}$, passing through P and Q, correspondingly. Notice that all verices must lie in the strip bordered by the lines p and q, as otherwise, $\overline{PQ}$ would not be the longest.

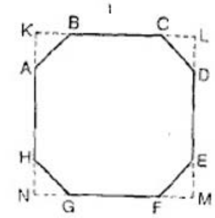


If PQ is a diagonal of the polygon, then choose the farthest (or one of the farthest) vertex X and Y on each side of this diagonal. Let x and y be the two lines parallel to $\overline{PQ}$, passing corespondingly through X and Y. Let $ABCD$ be the rectangle formed by lines p, q, x, and y. All vertices of the polygon must lie within this rectangle, by construction.

We have the following result $Area\ of\ ABCD = 2 \cdot area\ of\ PXQY \leq 2 \cdot area\ of\ the\ polygon$

If PQ is one of the sides of the polygon, one of the lines x or y are the same as PQ and the above reasoning holds as well.





5. First, notice that the area of the octagon from our problem is equal to the area of a cyclic octagon with alternating sides of length 1 and 2. By symmetry, all the interior angles are the same and their sum equals $(8-2)180° = 1080°$. Therefore, each interior angle is equal to $\frac{1080°}{8} = 135°$, and each exterior angle is equal to $45°$.

Consider extension of the octagon to the quadrilateral $KLMN$, as on the diagram. Since $\measuredangle KAB = \measuredangle KBA = 45°$ then $\measuredangle K = 90°$. Similarly, angles at L, M, N are all $90°$. So the quadrilateral is a square.

If $AB = 1$, then $AK = \frac{1}{\sqrt{2}}$. Similarly, all the arms of the right angle triangles are equal to $\frac{1}{\sqrt{2}}$. The area of the octagon equals to the area of the square $KLMN$ minus the area of the 4 right angle triangles. So, we have $\left(2 + \frac{2}{\sqrt{2}}\right)^2 - 2\left(\frac{1}{\sqrt{2}}\right)^2 = \left(2+\sqrt{2}\right)^2 - 1 = \mathbf{5 + 4\sqrt{2}}$

6. Group $1!\,2! \cdot 3!\,4! \cdot \ldots \cdot 99!\,100! = (1!\,3!\ \ldots\ 99!)^2 \cdot 2^{50} \cdot 50!$ If we remove **50!**, the remaining product is a perfect square of $1!\,3!\ \ldots\ 99! \cdot 2^{25}$.

7. If $n = 3k$ then $n^n = (n^k)^3$, hence n^n is a perfect cube. There are 100 natural numbers satisfying this condition.
If $n = 3k+1$, then $n^n = (n^k)^3 n$, which is a perfect cube if n is a perfect cube. If $n = 3k+2$, then $n^n = (n^k)^3 n^2$, which is a perfect cube if n is a perfect cube.
For $1 \leq n \leq 300$ there are four numbers of the form 3k +1 or 3k +2 which are perfect cubes (1, 8 $= 2^3$, 64 = 4^3, and 125 =5^3). So, together there are **104** integers n having the requested properties.

8. Using the given equation for $x = 0$, and $x = 2$, we obtain $P(0) = 0$ and $P(1) = 0$.

Therefore, $P(x)$ must have a form $x(x-1)Q(x)$, where $Q(x)$ satisfies the equation

$$x(x-1)(x-2)Q(x-1) = (x-2)x(x-1)Q(x)$$

So, we have $Q(x-1) = Q(x)$, which can happen only if $Q(x)$ is a constant polynomial. Thus $\boldsymbol{P(x) = cx(x-1)}$ for any constant c.