5^{th} Annual Math Match – February 11, 2014

1. The function $E(n)$ is defined for each positive integer n to be the sum of the even digits of n. For example, $E(5681) = 6 + 8 = 14$. What is the value of $E(1) + E(2) + \cdots + E(100)$?

2. Given a line l and points P, Q in a plane with l on opposite sides of l,
   a) determine a point R on l which maximixes $\big||PR| - |QR|\big|$,
   b) discuss existence of such point.

3. Let P be the incenter *(the center of the inscribed circle)* of a triangle ABC.
   Prove or disprove the statement:
   Each line passing through P divides the perimeter and the area of the triangle ABC in the same ratio.

4. Prove that $2^{\frac{1}{2}n(n-1)} > n!$ for any natural $n > 2$.

5. Determine whether or not there exists a function $f: \mathbb{N} \to \mathbb{N}$ such that $f(f(n)) = 2n$, for every $n \in \mathbb{N}$.

6. Consider a set of 4-digit numbers with the following property: the digits are all different and there is no “0” used. Find the sum of all such 4-digit numbers.

7. Start with an even number of points (at least four points) in the plane, no three on the same straight line, half colored blue and half colored yellow. Show there is a straight line, which does not meet any of the points, which divides the points into two non-empty sets of points, both sets being half blue and half yellow.

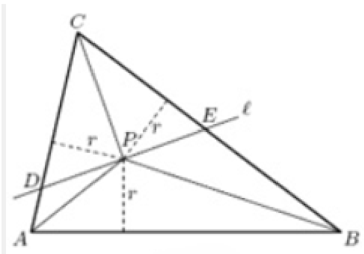
8. Let f be a function from the Euclidean plane $\mathbb{R}^2$ to $\mathbb{R}$ with the property: if A, B, C are the vertices of any triangle in $\mathbb{R}^2$, with circumcenter *(the center of the outscribed circle)* O, then $\frac{1}{3}\,[f(A) + f(B) + f(C)] = f(O)$. Show that f is constant.

Solutions:

1. Since the numbers $00, 01, 02, \dots, 99$ contain 20 of each digit, then $E(0) + E(1) + E)(2) + \cdots + E(99)$ equals $20(0 + 2 + 4 + 6 + 8) = 400$. Since $E(0) = 0$ and $E(100) = 0$, the required sum is also 400.

2. Reflect P over l, denoting the image by P'. Let R be any point on l. By triangle inequality, we have $\big||PR| - |QR|\big| = \big||P'R| - |QR|\big| \le |P'Q|$ with the equation maximizing the left side of the inequality. This occurs when P', Q, and R are collinear. Therefore, point R should be constructed as the intercept of $\overleftrightarrow{P'Q}$ with l. This can be done only if P and Q are not equidistant from l.

3. Let l be any line through the incenter P. WLOG assume that l crosses AC and BC at D and E correspondingly.



If r denotes the length of the radius of the incircle, we have

$$[ABED] = [DAP] + [ABP] + [BEP] = \tfrac{1}{2}DA \cdot r + \tfrac{1}{2}AB \cdot r + \tfrac{1}{2}BE \cdot r$$

$$= \tfrac{1}{2}(DA + AB + BE) \cdot r \quad \text{and}$$

$$[DEC] = [ECP] + [CDP] = \tfrac{1}{2}EC \cdot r + \tfrac{1}{2}CD \cdot r = \tfrac{1}{2}(EC + CD) \cdot r, \text{ so}$$

$\frac{[ABED]}{[DEC]} = \frac{DA+AB+BE}{EC+CD}$, that concludes the proof.

4. Proof by induction: 1° If $n = 3$, the inequality is true as $2^3 > 6$. 2° Assume the inequality is true for $k \ge 3$, then $2^{\frac{1}{2}(k+1)k} = 2^{\frac{1}{2}k(k-1)+k} = 2^{\frac{1}{2}k(k-1)} \cdot 2^k > k!\,2^k > k!\,(k+1) = (k+1)!$, so the inequality holds for all $n \ge 3$.

5. Let A be a set of positive integers such that 3 appears odd number of times in their prime factorization. Let $f(n) := \begin{cases} \frac{1}{3}n & \text{for} \quad n \in A \\ 6n & \text{for} \quad n \in \mathbb{N} \setminus A \end{cases}$. We have, if $n \in A$, then $f(n) \in \mathbb{N} \setminus A$, so $f\big(f(n)\big) = f\left(\frac{1}{3}n\right) = 2n$. Similarly, if $n \in \mathbb{N} \setminus A$, then $f(n) \in A$, so $f\big(f(n)\big) = f(6n) = 2n$.

6. The number of 4-digit numbers that can be made using digits from 1 to 9 without repetition is $9 \cdot 8 \cdot 7 \cdot 6$. Each number has its "complement" number such that the corresponding digits of the two numbers add to 10, for example 3562 and 7548. Since the number of such pairs is $\frac{1}{2} \cdot 9 \cdot 8 \cdot 7 \cdot 6$ and each pair adds to $10 \cdot (1000 + 100 + 10 + 1) = 11110$, then the required sum is $\frac{1}{2} \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 11110 = 16798320$.

7. Select any point P within the convex hull of the points, which is not one of the colored points and which is not on any line joining two of the colored points. Draw a line L through P which does not pass through any colored point. Attach a unit vector normal to L with its tail on L so as to indicate one of the two halves of the plane separated by L. Let $D = N(b) - N(y)$ be the number of blue points minus the number of yellow points in the indicated half plane. If D = 0, we are done. If not, rotate L counter clockwise around P by 180 degrees, causing D to change sign. Since during the rotation D changes in unit steps, there must have been some orientation in which $D = 0$. Because P is within the convex hull, each half plane is non-empty for any orientation. Thus the theorem is proved.

8. Consider any two points in the plane P and Q. Draw any circle thru P and Q. Let O be its center and A and B two other points on the circle. Then by the given $f(O) = \frac{1}{3}(f(A) + f(B) + f(P)) = \frac{1}{3}\big(f(A) + f(B) + f(Q)\big)$, so $f(P) = f(Q)$. Thus f is constant.